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Generalized Factorization for Daniele–Khrapkov Matrix Functions—Explicit Formulas

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Explicit formulas for the canonical generalized factorization of a class of matrix functions of the Daniele–Khrapkov class are derived. The method followed in the paper is a modification of that used for the investigation of conditions for the existence of canonical factorization proposed in a previous paper by the same authors. An example related to applications in diffraction theory is given. © 1995 Academic Press, Inc.

1. INTRODUCTION

In this paper the problem of calculating the factors of a canonical generalized factorization of a class of matrix functions of Daniele–Khrapkov type is solved. Specifically, matrix functions: $G\ \mathbb{R} \to \mathbb{C}^{2\times 2}$ of the form

$$G = \begin{bmatrix} a & b \\ \rho^2 b & a \end{bmatrix}, \tag{1.1}$$

where $\rho \in C(\dot{\mathbb{R}})$ is the square-root of two second degree polynomials are studied. The first degree case has received considerable attention in the literature and can be considered completely solved (see [7], [9], [10]). The paper is a follow up to [2] by the same authors. Having obtained conditions for the existence of canonical factorization in the first paper we turn now to the question of calculating the factors.

The problem addressed in this paper was also studied by other authors ([4], [5], [8], [9]) but truly explicit formulas were not obtained except for

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special cases, which is understandable since the question of existence of canonical factorization had not yet been completely answered.

The method followed in the subsequent sections is a modification of that used for the investigation of conditions of existence of canonical factorization proposed in the above-mentioned paper. It is shown that the solutions of the Riemann–Hilbert problem with coefficient G which give the factors of G are related by a simple recursive system of algebraic equations to the homogeneous Riemann–Hilbert problem corresponding to the calculation of the functions in the kernel of a Toeplitz operator whose symbol $\tilde{G}$ possesses a non-canonical factorization. The solutions to this other problem are easier to obtain and were actually calculated in the first paper. As a by-product of the analysis the authors solve explicitly an old problem stated by Daniele, that of determining a rational function R in the same group as G, such that the factors in the factorization of RG belong to this group. These factors are then given by simple formulas analogous to those for the case where ρ is the square-root of the quotient of first degree polynomials.

The paper is organized as follows. In Section 2 we give some results from the first paper which are needed for the sequel. In Sections 3 and 4 we present a method which leads to explicit formulas for the factors in a canonical generalized factorization. In Section 3 we consider in particular the case where ρ^2 does not admit canonical factorization, while in Section 4 we study the case where ρ^2 admits a canonical factorization. Finally, in Section 5, we apply the results established in the preceding sections to obtain the canonical factorization for an example in the class of symbols studied in this paper.

2. Preliminaries

As explained in the Introduction, the work developed in the present paper is closely related to the results obtained in [2]. We use here the same notation as in that paper and therefore we present in this section only some less usual notations and those results from [2] that are necessary for an independent reading of the present paper.

Related to $L_1^{\pm}(\mathbb{R})$ we define the spaces $B_1^{\pm}$ by

$$f \in B_1^{\pm} \Leftrightarrow r_{\pm} f \in L_1^{\pm}(\mathbb{R}) \tag{2.1}$$

(where $r_{\pm}(\xi) = (\xi \pm i)^{-1}$, for $\xi \in \mathbb{R}$). These spaces are such that (cf. [12]),

$$B_1^{+} \cap B_1^{-} = (0). \tag{2.2}$$

If A is an algebra, we denote by $\mathcal{G}A$ the group of invertible elements in A. Moreover, if $A \subset (L_\infty(\mathbb{R}))^{2\times 2}$ and $G \in A$ admits a generalized factorization $G = G_- \operatorname{diag}(r^{\mu_j})_{j=1}^2 G_+$ (where $r = r_+ r_-^{-1}$, cf. [2] for a precise definition) such that the factors belong to $\mathcal{G}A$, we say that this generalized factorization is a *factorization in the algebra A*. For the particular case where $A = (L_\infty(\mathbb{R}))^{2\times 2}$ a factorization in A will be called a *bounded factorization*.

Let I denote the identity matrix of order 2 and

$$R = \begin{bmatrix} 0 & 1 \\ \rho^2 & 0 \end{bmatrix}, \tag{2.3}$$

where ρ^2 is the quotient of two second degree polynomials,

$$\rho^2(\xi) = \frac{(\xi - k_1)(\xi - k_2)}{(\xi - k_3)(\xi - k_4)} \qquad (\xi \in \mathbb{R}), \tag{2.4}$$

k_j, $j = 1, 2, 3, 4$, being given (different) constants in $\mathbb{C}\backslash\mathbb{R}$. We denote by A_K the algebra of all matrix-valued functions of the form

$$G = a_1 I + a_2 R \tag{2.5}$$

With $a_1, a_2 \in C^\alpha(\dot{\mathbb{R}})$. These matrices are diagonalizable and we can write

$$G = HDH^{-1}, \tag{2.6}$$

where

$$H = \begin{bmatrix} 1 & 1 \\ \rho & -\rho \end{bmatrix}, \qquad D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \tag{2.7}$$

for $d_1 = a_1 + \rho a_2$, $d_2 = a_1 - \rho a_2$. Assuming that d_1 and d_2 admit a canonical bounded factorization $d_j = d_{j-} d_{j+}$ $(j = 1, 2)$, we will need in the sequel two results from [2], which follow. We remark, however, that, in order to make the procedure to obtain explicit formulas for the factors clearer, we consider a particular form for ρ^2. This has the advantage of greater computational simplicity without avoiding an understanding of how this method can be used to factorize more general matrix-functions of the same type. Therefore we present the following two theorems in the form that suits our case best.

THEOREM 2.1. *Let $k_1 = -i$, $k_2 = -2i$, $k_3 = i$, $k_4 = 2i$, in (2.4). Then G admits a canonical generalized factorization iff*

$$\int_{\mathbb{R}} \frac{\log(-d_1/d_2)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi \neq 0. \tag{2.8}$$

If condition (2.8) is not satisfied, the homogeneous equation $G\phi^+ = \phi^-$ admits non-trivial solutions

$$\phi^+ = (\phi_1^+, \phi_2^+), \qquad \phi^- = (\phi_1^-, \phi_2^-) \tag{2.9}$$

with

$$\phi_1^+ = \sqrt{-\frac{C d_{1+}^{-1} d_{2+}^{-1}}{(\xi+i)(\xi+2i)}}\ \mathrm{ch}\left(\frac{1}{2} r_+^{-1} \rho^{-1} F^+\right) \tag{2.10}$$

$$\phi_2^+ = -\sqrt{-\frac{C d_{1+}^{-1} d_{2+}^{-1}}{(\xi+i)(\xi+2i)}}\ \rho\ \mathrm{sh}\left(\frac{1}{2} r_+^{-1} \rho^{-1} F^+\right) \tag{2.11}$$

$$\phi_1^- = \sqrt{\frac{C d_{1-} d_{2-}}{(\xi-i)(\xi-2i)}}\ \rho^{-1}\ \mathrm{sh}\left(\frac{1}{2} r_+^{-1} \rho^{-1} F^-\right) \tag{2.12}$$

$$\phi_2^- = \sqrt{\frac{C d_{1-} d_{2-}}{(\xi-i)(\xi-2i)}}\ \mathrm{ch}\left(\frac{1}{2} r_+^{-1} \rho^{-1} F^-\right), \tag{2.13}$$

where C is a constant in $\mathbb{C}\backslash\{0\}$ and $F^\pm = P^\pm F$ for

$$F = r_+ \rho \log(-d_1/d_2). \tag{2.14}$$

In this case G admits a (non-canonical) generalized factorization with partial indices $\mu_1 = -1$, $\mu_2 = 1$.

THEOREM 2.2. *Let $k_1 = -2i$, $k_2 = i$, $k_3 = 2i$, $k_4 = -i$ in (2.4). Then G admits a canonical generalized factorization iff*

$$\int_{\mathbb{R}} \frac{\log\left(\dfrac{d_1}{d_2}\dfrac{(\sqrt{2/3}-\rho_-)(\sqrt{3/2}-\rho_+)}{(\sqrt{2/3}+\rho_-)(\sqrt{3/2}+\rho_+)}\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi \neq 0, \tag{2.15}$$

where

$$\rho_-(\xi) = \sqrt{\frac{\xi - i}{\xi - 2i}}, \qquad \rho_+(\xi) = \sqrt{\frac{\xi + 2i}{\xi + i}}. \tag{2.16}$$

3. Canonical Generalized Factorization—Explicit Formulas: First Case

Let $G = aI + bR \in A_K$ where R is defined as in (2.3) and, considering for simplicity a particular case,

$$\rho^2(\xi) = \frac{(\xi + i)(\xi + 2i)}{(\xi - i)(\xi - 2i)}, \qquad \text{for } \xi \in \mathbb{R}. \tag{3.1}$$

We assume in this section that the eigenvalues d_1, d_2 admit a bounded canonical factorization and that G also admits a canonical generalized factorization, i.e., condition (2.8) is satisfied, and we look for explicit formulas for the factors G_-, G_+ in a canonical factorization of G.

These will be obtained from the solutions of a system of equations, according to the following theorem.

THEOREM 3.1. *Let G admit a canonical generalized factorization and let $\Phi^+ \in (L_2^+(\mathbb{R}))^{2\times 2}$, $\Phi^- \in (L_2^-(\mathbb{R}))^{2\times 2}$ be such that*

$$G\Phi^+ = r\Phi^- \tag{3.2}$$

with $r(\xi) = (\xi - i)/(\xi + i)$, for $\xi \in \mathbb{R}$. Then a canonical generalized factorization of G is $G = G_-G_+$, with

$$G_- = r_-^{-1}\Phi^-, \qquad G_+^{-1} = r_+^{-1}\Phi^+, \tag{3.3}$$

if $\Phi^-(-i)$ is invertible (or $\Phi^+(i)$ is invertible).

Proof. Let $G = \tilde{G}_-\tilde{G}_+$ be a canonical generalized factorization. Then we have, from (3.2), assuming that $\tilde{G}_-(-i) = I$,

$$\tilde{G}_+\Phi^+ + 2ir_+\Phi^-(-i) = r, \qquad \tilde{G}_-^{-1}\Phi^- + 2ir_+\Phi^-(-i). \tag{3.4}$$

Since the left-hand side of (3.4) is in $(B_1^+)^{2\times 2}$, while the right-hand side is in $(B_1^-)^{2\times 2}$, it follows that both are zero; i.e.,

$$\tilde{G}_+\Phi^+ = -2ir_+\Phi^-(-i), \qquad \tilde{G}_-^{-1}\Phi^- = -2ir_-\Phi^-(-i).$$

Therefore, $\Phi^+ = -2ir_+\tilde{G}_+^{-1}\Phi^-(-i)$, $\Phi^- = -2ir_-\tilde{G}_-\Phi^-(-i)$ and we see, from the invertibility of $\Phi^-(-i)$, that Φ^+ and Φ^- are invertible and $G = G_-G_+$, with G_-, G_+ defined by (3.3), is a canonical generalized factorization of G. The proof is analogous if $\Phi_+(i)$ is invertible. ∎

Writing $\Phi^+ = (\Phi_1^+, \Phi_2^+)$, $\Phi^- = (\Phi_1^-, \Phi_2^-)$, it is clear that if Φ^+, Φ^- satisfy (3.2) then we also have

$$G\Phi_j^+ = r\Phi_j^-, \qquad j = 1, 2. \tag{3.5}$$

Therefore, we will obtain the two columns in G_- and G_+^{-1} separately, by solving two equations in $(L_2(\mathbb{R}))^2$, of the form (3.5), and showing that the solutions obtained are such that $\Phi^-(-i)$ is invertible.

Let

$$\varepsilon = \frac{\displaystyle\int_{\mathbb{R}} \frac{\log(-d_1/d_2)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi}{\displaystyle\int_{\mathbb{R}} \frac{1}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi}, \tag{3.6}$$

which, in the case where d_1/d_2 is constant, reduces to $\varepsilon = \log(-d_1/d_2)$ (where we assume that $\operatorname{Im}\log(-d_1/d_2) \leq 0$).

The necessary and sufficient condition for existence of a canonical factorization, presented in the preceding section, can thus be expressed by

$$\varepsilon \neq 0. \tag{3.7}$$

In the present section we assume first that condition (3.7) is satisfied, i.e., that G admits a canonical factorization, and we look for explicit formulas to obtain it.

The following theorem establishes conditions for the factors to belong to the algebra A_K, as well as their inverses, presenting also explicit formulas to determine these factors.

THEOREM 3.2. *G admits a canonical factorization $G = G_-G_+$ with $G_-^{\pm 1} \in A_K$, $G_+^{\pm 1} \in A_K$ iff $\varepsilon = -i\pi$, i.e.,*

$$\int_{\mathbb{R}} \frac{\log(d_1/d_2)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi = 0. \tag{3.8}$$

In this case $G = G_-G_+$ with

$$G_- = r_-^{-1}(\phi_1^- I + \rho^{-2}\phi_2^- R), \qquad G_+^{-1} = r_+^{-1}(\phi_1^+ I + \rho^{-2}\phi_2^+ R) \tag{3.9}$$

$$\phi_1^- = \sqrt{d_{1-}d_{2-}}\, r_-\, \mathrm{ch}(\tfrac{1}{2} r_+^{-1}\rho^{-1}F^-), \qquad \phi_2^- = \sqrt{d_{1-}d_{2-}}\, r_-\rho\, \mathrm{sh}(\tfrac{1}{2} r_+^{-1}\rho^{-1}F^-) \tag{3.10}$$

$$\phi_1^+ = \sqrt{d_{1+}^{-1}d_{2+}^{-1}}\, r_+\, \mathrm{ch}(\tfrac{1}{2} r_+^{-1}\rho^{-1}F^+), \qquad \phi_2^+ = -\sqrt{d_{1+}^{-1}d_{2+}^{-1}}\, r_+\rho\, \mathrm{sh}(\tfrac{1}{2} r_+^{-1}\rho^{-1}F^+) \tag{3.11}$$

for $F^\pm = P^\pm F$, $F = r_+\rho\log(d_1/d_2)$.

Proof. Let us begin by showing that (3.8) is necessary. Assume that $G = G_- G_+$ where $G_-^{\pm 1} \in A_K$, $G_+^{\pm 1} \in A_K$ and let us consider the first column in $r_- G_-$ and the first column in $r_+ G_+^{-1}$, which we denote by $\phi^- = [\phi_1^- \phi_2^-]^T$, $\phi^+ = [\phi_1^+ \phi_2^+]^T$, respectively. From the assumption G_+^{-1}, $G_- \in A_K$ it follows that

$$\phi_2^-(-i) = \phi_2^-(-2i) = 0, \qquad \phi_1^-(-i) \neq 0, \qquad \phi_1^-(-2i) \neq 0. \quad (3.12)$$

Now, (ϕ^+, ϕ^-) is the (unique) solution of the equation

$$G\phi^+ = r\phi^-, \quad (3.13)$$

with $\phi_-(-i) = (-1/2i, 0)$. Therefore,

$$\begin{cases} d_1(\phi_1^+ + \rho^{-1}\phi_2^+) = r(\phi_1^- + \rho^{-1}\phi_2^-) \\ d_2(\phi_1^+ - \rho^{-1}\phi_2^+) = r(\phi_1^- - \rho^{-1}\phi_2^-) \end{cases} \quad (3.14)$$

and, multiplying the left-hand sides and the right-hand sides of these two equations, we obtain

$$d_{1+}d_{2+}[(\phi_1^+)^2 - \rho^{-2}(\phi_2^+)^2] = d_{1-}^{-1}d_{2-}^{-1}r^2[(\phi_1^-)^2 - \rho^{-2}(\phi_2^-)^2]. \quad (3.15)$$

This yields

$$\begin{cases} d_{1+}d_{2+}[(\phi_1^+)^2 - \rho^{-2}\phi_2^+)^2] = \dfrac{p_1}{(\xi + i)^2(\xi + 2i)} \\ d_{1-}^{-1}d_{2-}^{-1}[(\phi_1^-)^2 - \rho^{-2}\phi_2^-)^2] = \dfrac{p_1}{(\xi - i)^2(\xi + 2i)}, \end{cases} \quad (3.16)$$

where p_1 is a polynomial whose degree is not greater than one. These equalities can also be expressed in the form

$$\begin{cases} d_{1+}d_{2+}[(\phi_1^+)^2 - \rho^{-2}\phi_2^+)^2] = \dfrac{p_1}{(\xi + i)^2(\xi + 2i)} \\ d_{1-}^{-1}d_{2-}^{-1}\left(\dfrac{\xi + 2i}{\xi - 2i}(\phi_1^-)^2 - \dfrac{\xi - i}{\xi + i}(\phi_2^-)^2\right) = \dfrac{p_1}{(\xi - i)^2(\xi - 2i)} \end{cases} \quad (3.17)$$

with (see (3.12))

$$\phi_2^- = \rho^2\tilde{\phi}_2^-, \ \tilde{\phi}_2^- \in L_2^-(\mathbb{R}). \quad (3.18)$$

Taking into account that $\phi_2^-(-2i) = 0$, it is clear from the second equality in (3.17) that $p_1(\xi) = \alpha(\xi + 2i)$ where $\alpha \neq 0$ is a constant. It follows from (3.18) that (3.16) is equivalent to

$$\begin{cases} d_{1+}d_{2+}[(\phi_1^+)^2 - \rho^{-2}(\phi_2^+)^2] = \dfrac{\alpha}{(\xi + i)^2} \\ d_{1-}^{-1}d_{2-}^{-1}[(\phi_1^-)^2 - \rho^{-2}(\tilde{\phi}_2^-)^2] = \dfrac{\alpha}{(\xi - i)^2}. \end{cases} \tag{3.19}$$

On the other hand, dividing the left-hand sides and the right-hand sides of the two equalities in (3.14), we also obtain

$$\frac{d_1}{d_2}\frac{\phi_1^+ + \rho^{-1}\phi_2^+}{\phi_1^+ - \rho^{-1}\phi_2^+} = \frac{\phi_1^- + \rho\tilde{\phi}_2^-}{\phi_1^- - \rho\tilde{\phi}_2^-}, \tag{3.20}$$

which yields

$$\log\frac{d_1}{d_2} + \log\frac{\phi_1^+ + \rho^{-1}\phi_2^+}{\phi_1^+ - \rho^{-1}\phi_2^+} = \log\frac{\phi_1^- + \rho\tilde{\phi}_2^-}{\phi_1^- - \rho\tilde{\phi}_2^-}. \tag{3.21}$$

Therefore

$$r_+\rho\log\frac{d_1}{d_2} + r_+\rho\log\frac{\phi_1^+ + \rho^{-1}\phi_2^+}{\phi_1^+ - \rho^{-1}\phi_2^+} = r\rho^2\left(r_-\rho^{-1}\log\frac{\phi_1^- + \rho\tilde{\phi}_2^-}{\phi_1^- - \rho\tilde{\phi}_2^-}\right) \tag{3.22}$$

(with $r\rho^2(\xi) = (\xi + 2i)/(\xi - 2i)$) and this yields, according to Lemma 3.1 in [2],

$$\begin{cases} r_+\rho\log\dfrac{\phi_1^+ + \rho^{-1}\phi_2^+}{\phi_1^+ - \rho^{-1}\phi_2^+} = -P^+\left(r_+\rho\log\dfrac{d_1}{d_2}\right) \\ r\rho^2\left(r_-\rho^{-1}\log\dfrac{\phi_1^- + \rho\tilde{\phi}_2^-}{\phi_1^- - \rho\tilde{\phi}_2^-}\right) = P^-\left(r_+\rho\log\dfrac{d_1}{d_2}\right). \end{cases} \tag{3.23}$$

The second equality in (3.23) now implies that we must have

$$P^-\left(r_+\rho\log\frac{d_1}{d_2}\right)(-2i) = 0, \tag{3.24}$$

which is equivalent to (3.8).

Conversely, if condition (3.8) is satisfied and if we define

$$F = r_+ \rho \log(d_1/d_2), \qquad F^{\pm} = P^{\pm}F, \tag{3.25}$$

then we have $F^-(-2i) = 0$ and it can be verified that a canonical factorization in the algebra A_K is $G = G_-G_+$ with G_- and G_+^{-1} given by the corresponding expressions enunciated in this theorem.

These expressions can be obtained from (3.23) and (3.19) and represent the solution of (3.14). ▌

Let us now assume that condition (3.8) is not satisfied, i.e., $\varepsilon \neq -i\pi$. To consider this case we begin by establishing some notation. Let

$$K = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-\frac{1}{4}u^2)}} > 0,$$

$$K' = -i\int_1^2 \frac{du}{\sqrt{(1-u^2)(1-\frac{1}{4}u^2)}} > 0,$$

$$\theta = K - \frac{\varepsilon}{\pi} K' \tag{3.26}$$

and let R_s denote the rectangle in the complex plane defined by

$$R_s = \{z \in \mathbb{C}\colon 0 < \operatorname{Re} z \leq K,\ 0 \leq \operatorname{Im} z \leq K'\}. \tag{3.27}$$

The condition of existence of a canonical factorization (3.7) corresponds, in terms of θ, to $\theta \neq K$, while condition (3.8) corresponds to $\theta = K + iK'$.

As explained above, we will now obtain the first column in G_- and G_+^{-1}. We start with some auxiliary results.

LEMMA 3.3. *If $\theta \in R_s$, $\theta \neq K, K + iK'$, then there exists $\mu \in \mathbb{C}$ such that*

$$\int_{\mathbb{R}} \frac{\log\left(-\dfrac{d_1}{d_2}\dfrac{1+\mu\rho r}{1-\mu\rho r}\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi = 0. \tag{3.28}$$

The value of μ satisfying (3.28) *is given by*

$$\mu = -\sqrt{\frac{(z_0-2i)(z_0+i)}{(z_0+2i)(z_0-i)}}, \qquad \textit{with } z_0 = -i \operatorname{sn}\theta, \tag{3.29}$$

where sn *denotes the Jacobian elliptic function* $\operatorname{sn}(u, \frac{1}{2})$ (*cf.* [11]).

Proof. We have

$$\int_{\mathbb{R}} \frac{\log\left(-\frac{d_1}{d_2}\frac{1+\mu\rho r}{1-\mu\rho r}\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi = 0. \tag{3.30}$$

$$\Leftrightarrow \int_{\mathbb{R}} \frac{\log\left(\frac{1+\mu\rho r}{1-\mu\rho r}\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi = -\int_{\mathbb{R}} \frac{\log\left(-\frac{d_1}{d_2}\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi$$

$$\Leftrightarrow \int_{\mathbb{R}} \frac{\log\left(\frac{1+\mu\rho r}{1-\mu\rho r}\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi = -\varepsilon \int_{\mathbb{R}} \frac{d\xi}{\sqrt{(\xi^2+1)(\xi^2+4)}}. \tag{3.31}$$

Let

$$G(\mu) = \int_{\mathbb{R}} \frac{\log\left(\frac{1+\mu\rho r}{1-\mu\rho r}\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi. \tag{3.32}$$

The function $1-\mu^2\rho^2 r^2$ has two zeros, z_0 and $2/z_0 (z_0 \in \mathbb{C}^-)$, such that $\mu^2 = [(z_0-2i)(z_0+i)]/[(z_0+2_i)(z_0-i)]$. We shall show that, for z_0 such that $\operatorname{Re} z_0 \geq 0$, $\operatorname{Im} z_0 < 0$, $z_0 \neq -2i$, and μ defined by (3.29), we have

$$G_1(z_0) = G(\mu(z_0)) = 2\pi i \int_{-i}^{z_0} \frac{du}{\sqrt{(u^2+1)(u^2+4}}. \tag{3.33}$$

In fact, from (3.32),

$$\begin{aligned} G'(\mu) &= \int_{\mathbb{R}} \frac{2\rho r}{\sqrt{(\xi^2+1)(\xi^2+4)}} \frac{1}{1-\mu^2\rho^2 r^2}\, d\xi \\ &= 2\int_{\mathbb{R}} \frac{d\xi}{(1-\mu^2)(\xi-z_0)(\xi-2/z_0)}, \end{aligned} \tag{3.34}$$

z_0 and $2/z_0$ being the zeros of $1-\mu^2\rho^2 r^2$. Since

$$1-\mu^2 = \frac{2iz_0}{(z_0+2i)(z_0-i)} \qquad (z_0 \neq -2i) \tag{3.35}$$

we can also write, from (3.34),

$$
\begin{aligned}
G'(\mu) &= -i\,\frac{(z_0 + 2i)(z_0 - i)}{z_0} \int_{\mathbb{R}} \frac{d\xi}{(\xi - z_0)(\xi - 2/z_0)} \\
&= 2\pi\,\frac{(z_0 + 2i)(z_0 - i)}{2 - z_0^2}.
\end{aligned} \tag{3.36}
$$

Equality (3.29) also yields

$$
\frac{d\mu}{dz_0} = i\,\frac{2 - z_0^2}{(z_0 + 2i)(z_0 - i)}\,\frac{1}{\sqrt{(z_0^2 + 1)(z_0^2 + 4)}} \tag{3.37}
$$

and it follows from (3.36) that

$$
G_1'(z_0) = \frac{2\pi i}{\sqrt{(z_0^2 + 1)(z_0^2 + 4)}}. \tag{3.38}
$$

Moreover, $G(\mu) = 0$ for $\mu = 0$; i.e., $G_1(z_0) = 0$ for $z_0 = -i$. Thus

$$
G_1(z_0) = 2\pi i \int_{-i}^{z_0} \frac{du}{\sqrt{(u^2 + 1)(u^2 + 4)}},
$$

as in (3.33). The meaning of the integral on the right-hand side of this equality is clear, since the function inside it is analytic in the interior of the domain defined for z_0 and continuous on its boundary. By the change of variable $x_0 = iz_0$, we obtain

$$
\begin{aligned}
G_2(x_0) = G_1(iz_0) &= \pi \int_1^{x_0} \frac{dv}{\sqrt{(1 - v^2)(1 - \frac{1}{4}v^2)}} \\
&= \pi \int_0^{x_0} \frac{dv}{\sqrt{(1 - v^2)(1 - \frac{1}{4}v^2)}} - K\pi
\end{aligned} \tag{3.39}
$$

for x_0 such that $\operatorname{Re} x_0 > 0$, $\operatorname{Im} x_0 \geq 0$.

Therefore, Eq. (3.31) can be reduced to

$$
G_2(x_0) = -\varepsilon \int_{\mathbb{R}} \frac{d\xi}{\sqrt{(\xi^2 + 1)(\xi^2 + 4)}} \tag{3.40}
$$

with $G_2(x_0)$ given by (3.39). The integral on the right-hand side of (3.40) can be expressed in terms of the constant K', using Cauchy's theorem.

We have

$$\begin{aligned}\int_{\mathbb{R}} \frac{d\xi}{\sqrt{(\xi^2+1)(\xi^2+4)}} &= -2\int_i^{2i} \frac{d\xi}{\sqrt{(\xi^2+1)(\xi^2+4)}} \\ &= -i\int_1^2 \frac{du}{\sqrt{(1-u^2)(1-\frac{1}{4}u^2)}} = K'.\end{aligned} \tag{3.41}$$

Therefore, (3.40) is equivalent to

$$\int_0^{x_0} \frac{du}{\sqrt{(1-u^2)(1-\frac{1}{4}u^2)}} = \theta \tag{3.42}$$

with θ defined by (3.26) and, for $\theta \in R_s$, we have $x_0 = \operatorname{sn}(K - \varepsilon K'/\pi)$ where sn denotes the Jacobian elliptic function $\operatorname{sn}(u, \frac{1}{2})$ (cf. [11]).

This yields $z_0 = -i \operatorname{sn} \theta$ and the value of μ that satisfies (3.28) is given by (3.29). ∎

Remark 3.4. It can be verified (cf. [6]) that, for $\varepsilon = -i\pi$ (which corresponds to condition (3.8)) we have $\theta = K + iK'$ and $x_0 = \operatorname{sn}(K + iK') = 2$, which implies that $z_0 = -2i$. Analogously, for $\varepsilon = 0$ (which corresponds to the case where G does not have a canonical factorization, see Theorem 2.1), we have $\theta = K$ and $x_0 = \operatorname{sn} K = 1$, which yields $z_0 = -i$.

LEMMA 3.5. *If $\theta \notin R_s$ and* Re $\varepsilon \geq 0$, *there exists $\mu \in \mathbb{C}$, $n \in \mathbb{N}$ such that*

$$\int_{\mathbb{R}} \frac{\log\left(-\frac{d_1}{d_2}\left(\frac{1+\mu\rho r}{1-\mu\rho r}\right)^n\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi = 0. \tag{3.43}$$

Proof. From (3.26),

$$\theta = \operatorname{Re}\theta + i\operatorname{Im}\theta = K - \frac{\operatorname{Re}\varepsilon}{\pi}K' - i\frac{\operatorname{Im}\varepsilon}{\pi}K', \tag{3.44}$$

which, in the particular case where d_1/d_2 is constant, can be expressed as

$$\theta = \left(K - \log\left|\frac{d_1}{d_2}\right|\frac{K'}{\pi}\right) - i\arg\left(-\frac{d_1}{d_2}\right)\frac{K'}{\pi}. \tag{3.45}$$

Assuming that Re $\varepsilon \geq 0$ and recalling that Im $\varepsilon \leq 0$ (as established at the beginning of this section), it is clear that there exists $n \in \mathbb{N}$ such that

$$\theta_n = K - \frac{\varepsilon}{n}\frac{K'}{\pi} = \left(K - \frac{\operatorname{Re}\varepsilon}{n}\frac{K'}{\pi}\right) - i\frac{\operatorname{Im}\varepsilon}{n}\frac{K'}{\pi} \tag{3.46}$$

satisfies the conditions $\operatorname{Re}\theta_n \in]0, K]$, $\operatorname{Im}\theta_n \in [0, K']$, i.e., $\theta_n \in R_s$. The determination of a convenient value for n is obvious.

Following the proof of Lemma 3.3, replacing ε by ε/n, we see that, for μ such that $z_0 = -i\,\mathrm{sn}(\theta_n)$ (cf. (3.29)),

$$\int_{\mathbb{R}} \frac{\log\left(\dfrac{1+\mu\rho r}{1-\mu\rho r}\right)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\,d\xi = -\frac{\varepsilon}{n}\int_{\mathbb{R}} \frac{d\xi}{\sqrt{(\xi^2+1)(\xi^2+4)}},$$

which yields (3.43).

Remark 3.6. We assumed in the previous lemma that $\operatorname{Re}\varepsilon \geq 0$ (which, in the case where d_1 and d_2 are constant, corresponds to assuming that $|d_1/d_2| \geq 1$). It can be easily seen that if $\operatorname{Re}\varepsilon \leq 0$ an analogous result holds, replacing n by $-n$ and considering $\operatorname{Im}\varepsilon \geq 0$ (instead of $\operatorname{Im}\varepsilon \leq 0$).

If $\tilde{Q}$ is a rational matrix belonging to A_K, then it must be of the form

$$\tilde{Q} = q_1 I + q_2 R, \qquad q_1, q_2 \in \mathcal{R}(\dot{\mathbb{R}}), \tag{3.47}$$

which is equivalent to

$$\tilde{Q} = H\begin{bmatrix} q_1 + \rho q_2 & 0 \\ 0 & q_1 - \rho q_2 \end{bmatrix} H^{-1}. \tag{3.48}$$

In particular we have, for $q_1 = 1$, $q_2 = \mu r$,

$$Q_1 = I + \mu r R = H\begin{bmatrix} 1 + \mu\rho r & 0 \\ 0 & 1 - \mu\rho r \end{bmatrix} H^{-1}. \tag{3.49}$$

The eigenvalues $1 \pm \mu\rho r$ admit a canonical bounded factorization for all μ such that $1 \pm \mu\rho r \in \mathcal{G}C(\dot{\mathbb{R}})$, which can be seen to mean that μ must be such that the zeros of $1 - \mu^2\rho^2 r^2$, z_0, and $2/z_0$ ($z_0 \in \mathbb{C}^-$), are not real. In particular, if $\theta_n \in R_s$ then $z_0 = -i\,\mathrm{sn}(\theta_n)$ is not real and the same happens with $2/z_0$.

Taking this into account and comparing the results of Lemmas 3.3 and 3.5 with the condition for existence of a canonical factorization (cf. Theorem 2.1), it is clear that we can interpret those results as follows.

PROPOSITION 3.7. *If G admits a canonical factorization and condition* (3.8) *is not satisfied, there exists a rational matrix Q_1 of the form* (3.49) *such that $Q_1^n G$ does not admit a canonical factorization. The space of solutions of the homogeneous equation $G_1 \phi_0^+ = \phi_0^-$ where $\phi_0^\pm \in (L_2^\pm(\mathbb{R}))^2$ and*

$$G_1 = Q_1^n G = H \operatorname{diag}(d_j')_{j=1}^2 H^{-1}, \tag{3.50}$$

$$d_1' = (1 + \mu\rho r)^n \, d_1, \qquad d_2' = (1 - \mu\rho r)^n \, d_2, \tag{3.51}$$

has dimension 1, *the solutions being given by formulas* (2.9)–(2.14) *with d_1 and d_2 replaced by d_1' and d_2', respectively.*

On the other hand, we know that the elements in the first column of $r_- G_-$, ϕ_1^- and ϕ_2^-, and in the first column of $r_+ G_+^{-1}$, ϕ_1^+, and ϕ_2^+, must satisfy (3.16), assuming that

$$\phi_1^-(-i) = -\frac{1}{2i} = K_0, \qquad \phi_2^-(-i) = 0. \tag{3.52}$$

Therefore $\phi^- = [\phi_1^- \phi_2^-]^T$ and $\phi^+ = [\phi_1^+ \phi_2^+]^T$ satisfy the equality $G\phi^+ = r\phi^-$, subject to (3.52), whence they also satisfy

$$Q_1^n G \phi^+ = r Q_1^n \phi^-. \tag{3.53}$$

For $n = 1$ we obtain from (3.53)

$$r^{-1} Q_1 G \phi^+ = (\phi^-)_1, \qquad (\phi^-)_1 \in (L_2^-(\mathbb{R}))^2 \tag{3.54}$$

(taking the second condition in (3.52) into account), with $(\phi^-)_1 = Q_1 \phi^- = [(\phi_1^-)_1 (\phi_2^-)_1]^T$ and

$$(\phi_2^-)_1(-i) = \mu(\rho^2 r \phi_1^-)_{(-i)} = -\frac{\mu}{3} K_0. \tag{3.55}$$

For $n = 2$ we analogously obtain

$$r^{-2} Q_1^2 G \phi^+ = (\phi^-)_2, \qquad (\phi^-)_2 \in (L_2^-(\mathbb{R}))^2, \tag{3.56}$$

with $(\phi^-)_2 = r^{-1} Q_1 (\phi^-)_1 = r^{-1} Q_1^2 \phi = [(\phi_1^-)_2 (\phi_2^-)_2]^T$ and

$$(\phi_1^-)_2(-i) = \mu(\phi_2^-)_1(-i) = -\frac{\mu^2}{3} K_0, \qquad (\phi_2^-)_2(-i) = 0. \tag{3.57}$$

In general we see that, if n is odd, we have

$$(\phi^-)_n = Q_1 (\phi^-)_{n-1} = r^{-k} Q_1^n \phi^- = [(\phi_1^-)_n (\phi_2^-)_n]^T \tag{3.58}$$

for $k = (n-1)/2$, with $(\phi_2^-)_n$ satisfying the condition

$$(\phi_2^-)_n(-i) = \frac{\mu^n}{(-3)^{(n+1)/2}} K_0. \tag{3.59}$$

If n is even, we obtain

$$(\phi^-)_n = r^{-1}Q_1(\phi^-)_{n-1} = r^{-k}Q_1^n\phi^- = [(\phi_1^-)_n(\phi_2^-)_n]^T \tag{3.60}$$

for $k = n/2$, subject to the conditions

$$(\phi_1^-)_n(-i) = \frac{\mu^n}{(-3)^{n/2}} K_0, \qquad (\phi_2^-)_n(-i) = 0. \tag{3.61}$$

Whether n is odd or even, it follows from $G\phi^+ = r\phi^-$ that $r^{-(k+1)}Q_1^nG\phi^+ = (\phi^-)_n$. This means that $(\phi^+, (\phi^-)_n)$ is a solution to the homogeneous equation

$$r^{-(k+1)}G_1\phi^+ = \phi^-. \tag{3.62}$$

Therefore, we now proceed to characterize the space of all the solutions of (3.62), beginning with an auxiliary result.

LEMMA 3.8. *A bounded (non-canonical) factorization of* G_1, *defined by* (3.50), *is given by*

$$G_1 = G_{1-}D\,G_{1+} \tag{3.63}$$

where

$$D = \operatorname{diag}(r^{-1}, r) \tag{3.64}$$

$$G_{1-} = \sqrt{d'_{1-}d'_{2-}}\begin{bmatrix} \rho_+ r\,\operatorname{sh}(\frac{1}{2}r_+^{-1}\rho^{-1}F^-) & \rho_-\operatorname{ch}(\frac{1}{2}r_+^{-1}\rho^{-1}F^-) \\ \rho_-^{-1}\operatorname{ch}(\frac{1}{2}r_+^{-1}\rho^{-1}F^-) & \rho_+^{-1}r^{-1}\operatorname{sh}(\frac{1}{2}r_+^{-1}\rho^{-1}F^-) \end{bmatrix}$$

$$G_{1+}^{-1} = \sqrt{-(d'_{1+}d'_{2+})^{-1}}\begin{bmatrix} \rho_+\operatorname{ch}(\frac{1}{2}r_+^{-1}\rho^{-1}F^+) & -\rho_- r\,\operatorname{sh}(\frac{1}{2}r_+^{-1}\rho^{-1}F^+) \\ -\rho_-^{-1}r^{-1}\operatorname{sh}(\frac{1}{2}r_+^{-1}\rho^{-1}F^+) & \rho_+^{-1}\operatorname{ch}(\frac{1}{2}r_+^{-1}\rho^{-1}F^+) \end{bmatrix}$$

and

$$\rho_+(\xi) = \sqrt{\frac{\xi+i}{\xi+2i}},\ \rho_-(\xi) = \sqrt{\frac{\xi-2i}{\xi-i}},\ F^\pm = P^\pm F,\ F = r_+\rho\log\left(-\frac{d'_1}{d'_2}\right).$$

Proof. Since $G_1 = H \operatorname{diag}(d_j')_{j=1}^2 H^{-1}$, with $d_j'(j = 1, 2)$ given by (3.51), satisfies the condition

$$\int_{\mathbb{R}} \frac{\log(-d_1'/d_2')(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi = 0 \tag{3.65}$$

it follows from Theorem 2.1 that G_1 does not admit a canonical factorization and, furthermore, a factorization of G_1 must be of the form (3.63) with D given by (3.64).

On the other hand, we have

$$G_0 = \begin{bmatrix} 0 & \rho^{-1} \\ \rho & 0 \end{bmatrix} = H \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} H^{-1} \tag{3.66}$$

and it follows from Theorem 3.2 and from (3.65) that $A = G_0 G_1$ admits a canonical factorization in the algebra A_K, $A = A_- A_+$, the factors being determined by the formulas (3.9)–(3.11).

Therefore, taking into account that $G_0 = G_0^{-1}$, we have $G_1 = A_- G_0 A_+ = (A_- G_{0-}) D (G_{0+} A_+) = G_{1-} D G_{1+}$ where

$$G_0 = G_{0-} D G_{0+} = \begin{bmatrix} 0 & \rho_- \\ \rho_-^{-1} & 0 \end{bmatrix} \begin{bmatrix} r^{-1} & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \rho_+^{-1} & 0 \\ 0 & \rho_+ \end{bmatrix}. \tag{3.67}$$

Using the formulas which give A_- and A_+^{-1} and the factors G_{0-} and G_{0+}^{-1} given by (3.67) we obtain the results stated in this lemma. ∎

COROLLARY 3.9. *A factorization of the matrix-valued function $r^{-(k+1)} G_1$ is given by*

$$r^{-(k+1)} G_1 = G_{1-} \begin{bmatrix} r^{-k-2} & 0 \\ 0 & r^{-k} \end{bmatrix} G_{1+}, \tag{3.68}$$

where G_{1-}, G_{1+} are defined in Lemma 3.8.

Now we can characterize the solutions of Eq. (3.62).

THEOREM 3.10. *A basis for the space of the solutions (ϕ^+, ϕ^-) to equation* (3.62), *with $k \geq 0$, consists of*

$$r^j(\phi_0^+, r^{-(k+1)}\phi_0^-), \qquad j = 0, \ldots, k+1 \tag{3.69}$$

and

$$r^s(\psi_0^+, r^{-(k-1)}\psi_0^-), \qquad s = 0, \ldots, k-1, \ (k \geq 1), \tag{3.70}$$

where (*cf. Lemma* 3.8)

$$\phi_0^+ = r_+(g_{11}^+, g_{21}^+) = [\phi_{01}^+\phi_{02}^+]^T, \qquad \phi_0^- = r_-(g_{11}^-, g_{21}^-) = [\phi_{01}^-\phi_{02}^-]^T \tag{3.71}$$

$$\psi_0^+ = r_+(g_{12}^+, g_{22}^+) = [\psi_{01}^+\psi_{02}^+]^T, \qquad \psi_0^- = r_-(g_{12}^-, g_{22}^-) = [\psi_{01}^-\psi_{02}^-]^T. \tag{3.72}$$

Proof. This is an immediate consequence of Lemma 3.8 and Corollary 3.9 in this paper and of Theorem 2.2, VII, 2 in [3]. ∎

COROLLARY 3.11. *Any solution* (ϕ^+, ϕ^-) *to Eq.* (3.62) *is of the form*

$$\phi^+ = (A_0 + A_1 r + \cdots + A_{k+1}r^{k+1})\phi_0^+ + \lambda(B_0 + B_1 r + \cdots + B_{k-1}r^{k-1})\psi_0^+$$

$$\begin{aligned}\phi^- = {} & (A_0 r^{-(k+1)} + A_1 r^{-k} + \cdots + A_{k+1})\phi_0^- \\ & + \lambda(B_0 r^{-(k-1)} + B_1 r^{-(k-2)} + \cdots + B_{k-1})\psi_0^-,\end{aligned} \tag{3.73}$$

where $\lambda = 0$ *if* $k = 0$, $\lambda = 1$ *if* $k \geq 1$.

In particular, $(\phi^+, (\phi^-)_n)$ (see (3.58) and (3.60)) will be of the form indicated in Corollary 3.11, where the constants $A_j, j = 0, \ldots, k+1$ and B_s, $s = 0, \ldots, k-1$, must be determined according to $2k + 2$ conditions. These are established bearing in mind that, if n is odd, (3.59) must be satisfied and, if n is even, (3.61) must be satisfied; moreover, the expressions (3.58) and (3.60), which define $(\phi^-)_n$ in terms of ϕ^-, must be reversible. This means that we must be able to obtain, after reversion of formulas (3.58) and (3.60), expressions defining $\phi_1^-, \phi_2^- \in L_2^-(\mathbb{R})$.

In fact, we have, from (3.58) and (3.60), the reversion formulas

$$\begin{cases}(\phi^-)_{n-1} = Q_1^{-1}(\phi^-)_n, & \text{if } n \text{ is odd,} \\ (\phi^-)_{n-1} = rQ_1^{-1}(\phi^-)_n, & \text{if } n \text{ is even,}\end{cases} \tag{3.74}$$

and

$$\begin{cases}(\phi^-)_{n-1} = r^{(n-1)/2}Q_1^{-n}(\phi^-)_n, & \text{if } n \text{ is odd,} \\ (\phi^-)_{n-1} = r^{n/2}Q_1^{-n}(\phi^-)_n, & \text{if } n \text{ is even.}\end{cases} \tag{3.75}$$

It is easy to see that (3.74) means that

$$\begin{cases}(\phi_1^-)_{n-1} = \dfrac{(\phi_1^-)_n - \mu r(\phi_2^-)_n}{1 - \mu^2\rho^2 r^2}, \\[2ex] (\phi_2^-)_{n-1} = \dfrac{(\phi_2^-)_n - \mu\rho^2 r(\phi_1^-)_n}{1 - \mu^2\rho^2 r^2} = (\phi_2^-)_n - \mu\rho^2 r(\phi_1^-)_{n-1}\end{cases} \tag{3.76}$$

if n is odd;

$$\begin{cases} (\phi_1^-)_{n-1} = r \dfrac{(\phi_1^-)_n - \mu r (\phi_2^-)_n}{1 - \mu^2 \rho^2 r^2}, \\ (\phi_2^-)_{n-1} = r \dfrac{(\phi_2^-)_n - \mu \rho^2 r (\phi_1^-)_n}{1 - \mu^2 \rho^2 r^2} = r(\phi_2^-)_n - \mu\rho^2 r (\phi_1^-)_{n-1} \end{cases} \tag{3.77}$$

if n is even.

It is clear, since

$$(1 - \mu^2\rho^2 r^2)(\xi) = \frac{(1-\mu^2)(\xi - z_0)(\xi - 2/z_0)}{(\xi - 2i)(\xi + i)}, \qquad \xi \in \mathbb{R}, \tag{3.78}$$

with $z_0 \in \mathbb{C}^-$, that we must add to condition (3.59) and (3.61) certain conditions at the point $\xi = z_0$ in order to obtain, by (3.76) and (3.77), functions which belong to $L_2^-(\mathbb{R})$.

From (3.75) we obtain:

(i) if n is odd,

$$\phi_j^- = \frac{r^{(n-1)/2}}{(1 - \mu^2\rho^2 r^2)^n} \psi_j, \qquad j = 1, 2, \tag{3.79}$$

where

$$\begin{aligned} \psi_1 = {} & \left[\binom{n}{0} (\phi_1^-)_n - \mu r \binom{n}{1} (\phi_2^-)_n\right] \\ & + \left[\binom{n}{2} (\phi_1^-)_n - \mu r \binom{n}{3} (\phi_2^-)_n\right] \mu^2\rho^2 r^2 \\ & + \cdots + \left[\binom{n}{n-1} (\phi_1^-)_n - \mu r \binom{n}{n} (\phi_2^-)_n\right] \mu^{n-1}\rho^{n-1} r^{n-1} \end{aligned} \tag{3.80}$$

and

$$\begin{aligned} \psi_2 = {} & \left[\binom{n}{0} (\phi_2^-)_n - \mu \rho^2 r \binom{n}{1} (\phi_1^-)_n\right] \\ & + \left[\binom{n}{2} (\phi_2^-)_n - \mu\rho^2 r \binom{n}{3} (\phi_1^-)_n\right] \mu^2\rho^2 r^2 \\ & + \cdots + \left[\binom{n}{n-1} (\phi_2^-)_n - \mu\rho^2 r \binom{n}{n} (\phi_1^-)_n\right] \mu^{n-1}\rho^{n-1} r^{n-1}; \end{aligned} \tag{3.81}$$

(ii) if n is even,

$$\phi_j^- = \frac{r^{n/2}}{(1 - \mu^2\rho^2 r^2)^n} \psi_j, \qquad j = 1, 2, \tag{3.82}$$

where

$$\begin{aligned}\psi_1 = {} & \left[\binom{n}{0} (\phi_1^-)_n - \mu r \binom{n}{1} (\phi_2^-)_n\right] \\ & + \left[\binom{n}{2} (\phi_1^-)_n - \mu r \binom{n}{3} (\phi_2^-)_n\right] \mu^2\rho^2 r^2 \\ & + \cdots + \left[\binom{n}{n-2} (\phi_1^-)_n - \mu r \binom{n}{n-1} (\phi_2^-)_n\right] \mu^{n-2}\rho^{n-2} r^{n-2} \\ & + \binom{n}{n} (\phi_1^-)_n \mu^n \rho^n r^n \end{aligned} \tag{3.83}$$

and

$$\begin{aligned}\psi_2 = {} & \left[\binom{n}{0} (\phi_2^-)_n - \mu\rho^2 r \binom{n}{1} (\phi_1^-)_n\right] \\ & + \left[\binom{n}{2} (\phi_2^-)_n - \mu\rho^2 r \binom{n}{3} (\phi_1^-)_n\right] \mu^2\rho^2 r^2 \\ & + \cdots + \left[\binom{n}{n-2} (\phi_2^-)_n - \mu\rho^2 r \binom{n}{n-1} (\phi_1^-)_n\right] \mu^{n-2}\rho^{n-2} r^{n-2} \\ & + \binom{n}{n} (\phi_2^-)_n \mu^n \rho^n r^n. \end{aligned} \tag{3.84}$$

Therefore we must impose a zero of order n for $\xi = z_0$ to the function ψ_1 defined by (3.80) for n odd and by (3.83) for n even.

It is important to remark that this condition on ψ_1 is much simpler than it may seem. In fact it reduces to imposing a zero of order n for $\xi = z_0$ to a certain rational function, as we show next.

It is clear from (3.76) and (3.77) that if $(\phi_1^-)_{n-1} \in L_2^-(\mathbb{R})$ then $(\phi_2^-)_{n-1} \in L_2^-(\mathbb{R})$. This implies that if the expressions on the right-hand side of (3.79) and (3.82) (which can be recursively obtained from (3.76) and (3.77)) define functions in $L_2^-(\mathbb{R})$ for $j = 1$, then the same happens with these expressions for $j = 2$. Therefore, if ψ_1 has a zero of order n for $\xi = z_0$, then the same is true for ψ_2. Consequently $\rho^2\psi_1^2 - \psi_2^2$ must have a zero of order at least equal to $2n$ for $\xi = z_0$.

Now, using the relations

$$\psi_{01}^- = \rho_-^2\phi_{02}^-, \qquad \psi_{02}^- = \rho_+^{-2}r^{-2}\phi_{01}^-$$

as well as the expressions for $(\phi_1^-)_n$, $(\phi_2^-)_n$ established in Corollary 3.11, it can be verified that

$$\rho^2\psi_1^2 - \psi_2^2 = (1 - \mu^2\rho^2r^2)^n[\rho^2(\phi_{01}^-)^2 - (\phi_{02}^-)^2] \cdot Q,$$

where Q is a rational function of the form

$$\begin{aligned} Q = {} & (A_0r^{-(k+1)} + A_1r^{-k} + \cdots + A_{k+1})^2 \\ & - \lambda(B_0r^{-(k-1)} + B_1r^{-(k-2)} + \cdots + B_{k-1})\rho^2\rho_-^4. \end{aligned} \tag{3.85}$$

Since the function $1 - \mu^2\rho^2r^2$ has a simple zero for $\xi = z_0$ and $\rho^2(\phi_{01}^-)^2 - (\phi_{02}^-)^2$ never vanishes in $\mathbb{C}^-$, we see that $\rho^2\psi_1^2 - \psi_2^2$ has a zero of order $2n$ iff Q has a zero of order n for $\xi = z_0$.

The conclusions obtained so far can be summarized as follows:

THEOREM 3.12. *Let $n \in \mathbb{N}$, $\mu \in \mathbb{C}$ be such that* (3.43) *is satisfied and let $k = (n-1)/2$ if n is odd, $k = n/2$ if n is even. Let ϕ_0^+, ϕ_0^-, ψ_0^+, ψ_0^- be defined as in Theorem* 3.10 *and*

$$\begin{aligned} (\phi_1^+, \phi_2^+) = \phi^+ = {} & (A_0 + A_1r + \cdots + A_{k+1}r^{k+1})\phi_0^+ \\ & + \lambda(B_0 + B_1r + \cdots + B_{k-1}r^{k-1})\psi_0^+ \end{aligned} \tag{3.86}$$

$$\begin{aligned} ((\phi_1^-)_n, (\phi_2^-)_n) = (\phi^-)_n = {} & (A_0 + A_1r + \cdots + A_{k+1}r^{k+1})r^{-(k+1)}\phi_0^- \\ & + \lambda(B_0 + B_1r + \cdots + B_{k-1}r^{k-1})r^{-(k-1)}\psi_0^- \end{aligned} \tag{3.87}$$

with $\lambda = 0$ if $k = 0$, $\lambda = 1$ if $k \geq 1$.

Let furthermore A_j, $j = 0, \ldots, k+1$, B_s, $s = 0, \ldots, k-1$, be such that Q defined by (3.85) *has a zero of order n at $\xi = z_0$ and*

(i) *if n is odd,*

$$(\phi_2^-)_n(-i) = \frac{\mu^n}{(-3)^{(n+1)/2}} K_0; \tag{3.88}$$

(ii) *if n is even,*

$$(\phi_1^-)_n(-i) = \frac{\mu^n}{(-3)^{n/2}} K_0, \qquad (\phi_2^-)_n(-i) = 0. \tag{3.89}$$

Then the solution of (3.14) *satisfying conditions* (3.52) *is obtained by* (3.75). *In particular,* $\phi_j^-(j = 1, 2)$ *is given by* (3.79), *if n is odd, or by* (3.82), *if n is even.*

As we remarked earlier, ϕ^+ and ϕ^-, determined as in Theorem 3.12, provide the first columns of r_-G_- and $r_+G_+^{-1}$. To determine the second columns in r_-G_- and $r_+G_+^{-1}$ we take into account that the elements in these two columns must also satisfy (3.14). However, in this case, it will be convenient to impose certain conditions (corresponding to (3.52) for the first column) on the second column of $r_+G_+^{-1}$, instead of r_-G_-.

Thus we now consider the system of equations

$$\begin{cases} d_1(\phi_3^+ + \rho^{-1}\phi_4^+) = r(\phi_3^- + \rho^{-1}\phi_4^-) \\ d_2(\phi_3^+ - \rho^{-1}\phi_4^+) = r(\phi_3^- - \rho^{-1}\phi_4^-) \end{cases} \tag{3.90}$$

whose solutions must satisfy

$$\phi_3^+(i) = 0, \qquad \phi_4^+(i) = \frac{1}{2i} = -K_0. \tag{3.91}$$

First, however, we establish a condition to ensure that (3.90) and (3.91) lead us to a second column which is linearly independent from the column $[\phi_1^+ \phi_2^+]^T$ which was determined above as being the first column of $r_+G_+^{-1}$. In other words, we ensure that

$$G_+^{-1} = r_+^{-1} \begin{bmatrix} \phi_1^+ & \phi_3^+ \\ \phi_2^+ & \phi_4^+ \end{bmatrix}, \tag{3.92}$$

with ϕ_1^+, ϕ_2^+ determined as above and ϕ_3^+, ϕ_4^+ determined from (3.90) and (3.91), is such that $\det G_+^{-1}(i) \neq 0$ (see Theorem 3.1).

PROPOSITION 3.13. *Let* $(\phi_1^+, \phi_2^+, \phi_1^-, \phi_2^-)$ *be the solution of* (3.14) *satisfying condition* (3.52). *Then* $\phi_1^+(i) \neq 0$, *unless*

$$\int_{\mathbb{R}} \frac{\log\left(\dfrac{d_1}{d_2}\dfrac{(\sqrt{2/3} - \tilde{\rho}_-)}{(\sqrt{2/3} + \tilde{\rho}_-)}\dfrac{(\sqrt{3/2} - \tilde{\rho}_+)}{(\sqrt{3/2} + \tilde{\rho}_+)}\right)(\xi)}{\sqrt{(\xi^2 + 1)(\xi^2 + 4)}}\, d\xi = 0, \tag{3.93}$$

where

$$\tilde{\rho}_-(\xi) = \sqrt{\frac{\xi - i}{\xi - 2i}}, \qquad \tilde{\rho}_+(\xi) = \sqrt{\frac{\xi + 2i}{\xi + i}}.$$

Proof. Since $\phi_2^-(-i) = 0$, we have $\phi_2^- = r^{-1}\tilde{\phi}_2^-$, with $\tilde{\phi}_2^- \in L_2^-(\mathbb{R})$. Let us assume that $\phi_1^+(i) = 0$. Then we also have $\phi_1^+ = r\tilde{\phi}_1^+$, with $\tilde{\phi}_1^+ \in L_2^+(\mathbb{R})$. This implies that (cf. (3.14))

$$\begin{cases} d_1(\tilde{\phi}_1^+ + \tilde{\rho}^{-1}\phi_2^+) = \phi_1^- + \tilde{\rho}^{-1}\tilde{\phi}_2^- \\ d_2(\tilde{\phi}_1^+ - \tilde{\rho}^{-1}\phi_2^+) = \phi_1^- - \tilde{\rho}^{-1}\tilde{\phi}_2^-, \end{cases} \tag{3.94}$$

where $\tilde{\rho}^{-1} = r^{-1}\rho^{-1}$ admits a canonical generalized factorization. It is clear from Theorem 2.2 that (3.94) admits non-trivial solutions iff condition (3.93) is satisfied. ∎

Remark 3.14. The constants $\sqrt{2/3}$ and $\sqrt{3/2}$ in (3.93) can be replaced by α and β, respectively, provided that $\alpha^2 - \tilde{\rho}_-^2$ has (only) one zero in $\mathbb{C}^-$ and $\beta^2 - \tilde{\rho}_+^2$ has (only) one zero in $\mathbb{C}^+$.

Remark 3.15. The factorization of G can still be obtained using the same method which yields a factorization in the algebra A_K (see Theorem 3.2). In this case we impose $\phi_{1-}(-i) \neq 0$, $\phi_{2-}(-i) = 0$, $\phi_{1+}(i) = 0$ for the first column and $\phi_{3+}(2i) = 0$, $\phi_{4+}(2i) \neq 0$, $\phi_{4-}(-2i) = 0$ for the second column. The relations $\phi_{3+}(i) \neq 0$, $\phi_{4-}(-i) \neq 0$ appear as a consequence and so the conditions $\det G_+(i) \neq 0$, $\det G_-(-i) \neq 0$ are satisfied.

Now we determine the solution of (3.90), (3.91), following the same line of argument that we used to obtain the results of Theorem 3.12, beginning with an alternative form for (3.43).

LEMMA 3.16. *The following conditions are equivalent*:

$$\text{(i)} \quad \int_{\mathbb{R}} \frac{\log\left(-\frac{d_1}{d_2}\left(\frac{1 + \mu\rho r}{1 - \mu\rho r}\right)^n\right)(\xi)}{\sqrt{(\xi^2 + 1)(\xi^2 + 4)}}\, d\xi = 0$$

$$\text{(ii)} \quad \int_{\mathbb{R}} \frac{\log\left(-\frac{d_1}{d_2}\left(\frac{1 + \mu\rho^{-1}r^{-1}}{1 - \mu\rho^{-1}r^{-1}}\right)^n\right)(\xi)}{\sqrt{(\xi^2 + 1)(\xi^2 + 4)}}\, d\xi = 0.$$

Proof. We have $\rho r(-\xi) = \rho^{-1}r^{-1}(\xi)$. Thus, changing the variable ξ to $-\xi$,

$$\int_{\mathbb{R}} \frac{\log\left(\frac{1 + \mu\rho r}{1 - \mu\rho r}\right)(\xi)}{\sqrt{(\xi^2 + 1)(\xi^2 + 4)}}\, d\xi = \int_{\mathbb{R}} \frac{\log\left(\frac{1 + \mu\rho^{-1}r^{-1}}{1 - \mu\rho^{-1}r^{-1}}\right)(\xi)}{\sqrt{(\xi^2 + 1)(\xi^2 + 4)}}\, d\xi$$

from which we obtain

$$\int_{\mathbb{R}} \frac{\log\left(\frac{1+\mu\rho r}{1-\mu\rho r}\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\,d\xi = -\frac{1}{n}\int_{\mathbb{R}} \frac{\log\left(-\frac{d_1}{d_2}\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\,d\xi$$

$$\Leftrightarrow \int_{\mathbb{R}} \frac{\log\left(\frac{1+\mu\rho^{-1}r^{-1}}{1-\mu\rho^{-1}r^{-1}}\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\,d\xi = -\frac{1}{n}\int_{\mathbb{R}} \frac{\log\left(-\frac{d_1}{d_2}\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\,d\xi.$$

∎

As we did in Proposition 3.7, we can use this result in order to relate the solution of (3.90) to the solutions of a homogeneous system of equations.

PROPOSITION 3.17. *If G admits a canonical factorization and condition* (3.8) *is not satisfied, there exists a rational matrix Q_2 of the form*

$$Q_2 = I - \mu\rho^{-2}r^{-1}R \tag{3.95}$$

such that GQ_2^{-n} admits a non-canonical factorization.

The space of solutions of the homogeneous equation

$$G_2\phi_0^+ = \phi_0^-, \tag{3.96}$$

where $\phi_0^\pm \in (L_2^\pm(\mathbb{R}))^2$ and

$$G_2 = GQ_2^{-n} = H\,\mathrm{diag}\,(d_j'')_{j=1}^2 H^{-1}, \tag{3.97}$$

$$d_1'' = (1-\mu\rho^{-1}r^{-1})^{-n}d_1, \qquad d_2'' = (1+\mu\rho^{-1}r^{-1})^{-n}d_2, \tag{3.98}$$

has dimension 1, *the solutions being given by formulas* (2.9)–(2.14) *with d_1 and d_2 replaced by d_1'' and d_2'', respectively.*

Now, the solutions of (3.90) also satisfy

$$r^{-1}GQ_2^{-n}(Q_2^n\phi^+) = \phi^- \tag{3.99}$$

with $\phi^+ = [\phi_3^+\ \phi_4^+]^T$, $\phi^- = [\phi_3^-\ \phi_4^-]^T$. Since $Q_2^n\phi^+ = r^{-k}(\phi^+)_n$ where $(\phi^+)_n = [(\phi_3^+)_n(\phi_4^+)_n]^T \in (L_2^+)(\mathbb{R}))^2$ and

(i) if n is odd, $k = (n-1)/2$,

$$(\phi^+)_n = Q_2(\phi^+)_{n-1}, \tag{3.100}$$

$$(\phi_3^+)_n(i) = -\mu(\rho^{-2}r^{-1})(i)(\phi_4^+)_{n-1}(i) = \frac{\mu^n}{(-3)^{(n+1)/2}}K_0; \tag{3.101}$$

(ii) if n is even, $k = n/2$,

$$(\phi^+)_n = rQ_2(\phi^+)_{n-1} \tag{3.102}$$

$$(\phi_3^+)_n(i) = 0,\ (\phi_4^+)_n(i) = -\mu(\phi_3^+)_{n-1}(i) = -\frac{\mu^n}{(-3)^{n/2}} K_0. \tag{3.103}$$

Thus from (3.99) we obtain $r^{-(k+1)}GQ_2^{-n}(\phi^+)_n = \phi^-$, which shows that $((\phi^+)_n, \phi^-)$ is a solution to the homogeneous equation

$$r^{-(k+1)}\, G_2\phi^+ = \phi^-. \tag{3.104}$$

The space of all solutions of (3.104) is characterized next.

LEMMA 3.18. *A bounded (non-canonical) factorization of the matrix function G_2, defined by* (3.97), *is given by*

$$G_2 = G_{2-}DG_{2+}, \tag{3.105}$$

where $D = \operatorname{diag}(r^{-1}, r)$, $G_{2-} = [g_{ij}^-]$, $G_{2+}^{-1} = [g_{ij}^+]$, and g_{ij}^-, g_{ij}^+ $(i, j = 1, 2)$ are defined as in Lemma 3.8 *with d_1', d_2' replaced by d_1'', d_2'', respectively.*

COROLLARY 3.19. *A factorization of the matrix-valued function $r^{-(k+1)}G_2$ is given by*

$$r^{-(k+1)}G_2 = G_{2-}\operatorname{diag}(r^{-(k+2)}, r^{-k})G_{2+} \tag{3.106}$$

with G_{2-}, G_{2+} defined in Lemma 3.18.

THEOREM 3.20. *A basis for the space of the solutions (ϕ^+, ϕ^-) to Eq.* (3.104), *with $k \geq 0$, consists of the functions defined in* (3.69)–(3.78), *for g_{ij}^- and g_{ij}^+ $(i, j = 1, 2)$ as in Lemma* 3.18.

Any solution (ϕ^+, ϕ^-) to Eq. (3.104) *is of the form* (3.73).

From (3.100) and (3.102) we obtain (as in the case of the first columns) the reversion formulas

$$\begin{cases} (\phi_3^+)_{n-1} = \dfrac{(\phi_3^+)_n + \mu\rho^{-2}r^{-1}(\phi_4^+)_n}{1 - \mu^2\rho^{-2}r^{-2}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , \quad \text{if } n \text{ is odd,} \\ (\phi_4^+)_{n-1} = \dfrac{(\phi_4^+)_n + \mu r^{-1}(\phi_3^+)_n}{1 - \mu^2\rho^{-2}r^{-2}} = \mu r^{-1}(\phi_3^+)_{n-1} + (\phi_4^+)_n \end{cases} \tag{3.107}$$

$$\begin{cases} (\phi_3^+)_{n-1} = r^{-1}\dfrac{(\phi_3^+)_n + \mu\rho^{-2}r^{-1}(\phi_4^+)_n}{1 - \mu^2\rho^{-2}r^{-2}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , \qquad \text{if } n \text{ is even}, \\ (\phi_4^+)_{n-1} = r^{-1}\dfrac{(\phi_4^+)_n + \mu r^{-1}(\phi_3^+)_n}{1 - \mu^2\rho^{-2}r^{-2}} = r^{-1}[\mu(\phi_3^+)_{n-1} + (\phi_4^+)_n] \end{cases} \tag{3.108}$$

where

$$(1 - \mu^2\rho^{-2}r^{-2})(\xi) = \frac{(1 - \mu^2)(\xi + z_0)(\xi + 2/z_0)}{(\xi + 2i)(\xi - i)}, \qquad \xi \in \mathbb{R}. \tag{3.109}$$

To guarantee that these reversion formulas lead to the determination of ϕ_3^+, $\phi_4^+ \in L_2^+(\mathbb{R})$ we will have to satisfy, as above, certain conditions relative to the point $-z_0 \in \mathbb{C}^+$ (as well as (3.101) or (3.103)).

We have

(i) if n is odd,

$$\phi_j^+ = \frac{r^{-(n-1)/2}}{1 - \mu^2\rho^{-2}r^{-2})^n}\,\psi_j, \qquad j = 3, 4, \tag{3.110}$$

where

$$\begin{aligned} \psi_3 = {} & \left[\binom{n}{0}(\phi_3^+)_n + \mu r^{-1}\rho^{-2}\binom{n}{1}(\phi_4^+)_n\right] \\ & + \left[\binom{n}{2}(\phi_3^+)_n + \mu r^{-1}\rho^{-2}\binom{n}{3}(\phi_4^+)_n\right]\mu^2\rho^{-2}r^{-2} \\ & + \cdots + \left[\binom{n}{n-1}(\phi_3^+)_n + \mu r^{-1}\rho^{-2}\binom{n}{n}(\phi_4^+)_n\right]\mu^{n-1}\rho^{-(n-1)}r^{-(n-1)} \end{aligned} \tag{3.111}$$

and

$$\begin{aligned} \psi_4 = {} & \left[\binom{n}{0}(\phi_4^+)_n + \mu r^{-1}\binom{n}{1}(\phi_3^+)_n\right] \\ & + \left[\binom{n}{2}(\phi_4^+)_n + \mu r^{-1}\binom{n}{3}(\phi_3^+)_n\right]\mu^2\rho^{-2}r^{-2} \\ & + \cdots + \left[\binom{n}{n-1}(\phi_4^+)_n + \mu r^{-1}\binom{n}{n}(\phi_3^+)_n\right]\mu^{n-1}\rho^{-(n-1)}r^{-(n-1)}; \end{aligned}$$

(ii) if n is even,

$$\phi_j^+ = \frac{r^{-n/2}}{(1 - \mu^2\rho^{-2}r^{-2})^n}\,\psi_j, \qquad j = 3, 4, \tag{3.112}$$

where

$$\begin{aligned}\psi_3 = {} & \left[\binom{n}{0}(\phi_3^+)_n + \mu r^{-1}\rho^{-2}\binom{n}{1}(\phi_4^+)_n\right] \\ & + \left[\binom{n}{2}(\phi_3^+)_n + \mu r^{-1}\rho^{-2}\binom{n}{3}(\phi_4^+)_n\right]\mu^2\rho^{-2}r^{-2} + \cdots \\ & + \left[\binom{n}{n-2}(\phi_3^+)_n + \mu r^{-1}\rho^{-2}\binom{n}{n-1}(\phi_4^+)_n\right]\mu^{n-2}\rho^{-(n-2)}r^{-(n-2)} \\ & + \binom{n}{n}(\phi_3^+)_n\mu^n\rho^{-n}r^{-n}\end{aligned} \tag{3.113}$$

and

$$\begin{aligned}\psi_4 = {} & \left[\binom{n}{0}(\phi_4^+)_n + \mu r^{-1}\binom{n}{1}(\phi_3^+)_n\right] \\ & + \left[\binom{n}{2}(\phi_4^+)_n + \mu r^{-1}\binom{n}{3}(\phi_3^+)_n\right]\mu^2\rho^{-2}r^{-2} \\ & + \cdots + \left[\binom{n}{n-2}(\phi_4^+)_n + \mu r^{-1}\binom{n}{n-1}(\phi_3^+)_n\right]\mu^{n-2}\rho^{-(n-2)}r^{-(n-2)} \\ & + \binom{n}{n}(\phi_4^+)_n\mu^n\rho^{-n}r^{-n}.\end{aligned}$$

Therefore, we must impose a zero of order n for $\xi = -z_0$ to the function ψ_3 defined by (3.111), for n odd, and by (3.113), for n even. As above this corresponds to imposing a zero of order n for $\xi = -z_0$ to the rational function

$$Q = (A_0 + A_1 r + \cdots + A_{k+1}r^{k+1})^2 - \lambda(B_0 + B_1 r + \cdots + B_{k-1}r^{k-1})^2\rho_+^{-4}\rho^{-2}. \tag{3.114}$$

We summarize in the next theorem the conclusions that we obtained, concerning the second column.

THEOREM 3.21. *With the same assumptions as in Theorem* 3.12 *and* d_1', d_2' *replaced by* d_1'', d_2'', *respectively, let* A_j $(j = 0, \ldots, k + 1)$, B_s

($s = 0, \dots, k-1$) *be such that* Q, *defined by* (3.114), *has a zero of order* n *for* $\xi = -z_0$ *and*

(i) *if n is odd*, (3.101) *is satisfied*;

(ii) *if n is even*, (3.103) *is satisfied.*

Then the solution of (3.90) *satisfying conditions* (3.91) *can be obtained from Corollary* 3.11. *In particular*, ϕ_j^+ ($j = 3, 4$) *is given by* (3.110), *if n is odd, or by* (3.112), *if n is even.*

We end this section with an interpretation of the results previously obtained in terms of the solution of a well-known problem, stated by Daniele in [4]. It consists of determining a rational function $\tilde{Q}$ of the form (3.47) such that $\tilde{Q}G$ admits a factorization in the algebra A_K. If such a rational matrix can be determined, we have

$$G_A = \tilde{Q}G = (G_A)_-(G_A)_+,$$

where $(G_A)_-^{\pm 1} \in (L_\infty^-(\mathbb{R}))^{2\times 2} \cap A_K$, $(G_A)_+^{\pm 1} \in (L_\infty^+(\mathbb{R}))^{2\times 2} \cap A_K$. Therefore,

$$G = (G_A)_- \, \tilde{Q}^{-1}(G_A)_+$$

and a generalized factorization for G can be determined by rational factorization of $\tilde{Q}^{-1}$.

In fact, in Lemmas 3.3 and 3.5 we establish that if G does not admit a factorization in the algebra A_K, there is a $\mu \in \mathbb{C}$ (which is explicitly determined in the proof) such that

$$\int_{\mathbb{R}} \frac{\log\left(-\dfrac{d_1}{d_2}\left(\dfrac{1+\mu\rho r}{1-\mu\rho r}\right)^n\right)(\xi)}{\sqrt{(\xi^2+1)(\xi^2+4)}}\, d\xi = 0. \tag{3.115}$$

For $r_1(\xi) = (\xi - 2i)/(\xi + 2i)$, we have $r_1\rho = (\rho r)^{-1}$; therefore

$$-\frac{1+\mu\rho r}{1-\mu\rho r} = \frac{\mu+\rho r_1}{\mu-\rho r_1}.$$

Thus,

$$-\left(\frac{1+\mu\rho r}{1-\mu\rho r}\right)^n = \left(\frac{\mu+\rho r_1}{\mu-\rho r_1}\right)^n, \qquad \text{if } n \text{ is odd,}$$

$$-\left(\frac{1+\mu\rho r}{1-\mu\rho r}\right)^n = \frac{\mu+\rho r_1}{\mu-\rho r_1}\left(\frac{1+\mu\rho r}{1-\mu\rho r}\right)^{n-1}, \qquad \text{if } n \text{ is even.}$$

Since $\mu \pm \rho r_1$ are the eigenvalues of $\tilde{Q}_1 = \mu I + r_1 R$ and $1 \pm \mu\rho r$ are the eigenvalues of $Q_1 = I + \mu r R$, it is clear from Theorem 3.2 that (3.115) means that if n is odd, $\tilde{Q}_1^n G$ admits a canonical factorization in the algebra A_K and if n is even, $\tilde{Q}_1 Q_1^{n-1} G$ admits a factorization of that type.

This answers Daniele's question, although it is obvious that the rational matrix in question is not unique and there may be other answers.

4. Canonical Generalized Factorization—Explicit Factorization: Second Case

Let us now consider the canonical generalized factorization of $G = aI + bR$ in a case where ρ^2 admits a canonical generalized factorization. Let

$$\rho^2(\xi) = \frac{(\xi - 2i)(\xi + 2i)}{(\xi - i)(\xi + i)}. \tag{4.1}$$

Although the method presented in the preceding section can be applied in this case to obtain the factors in a canonical generalized factorization of G (in the case where the factors are in the same algebra, as well as in the case where they are not), we avoid repetition by presenting this problem in a new perspective.

For ρ^2 defined as in (4.1), let A_K be the algebra of all matrix-valued functions of the form $G = aI + bR$, where $a, b \in C^\alpha(\dot{\mathbb{R}})$ and R is defined by (2.3).

We denote by $\tilde{A}_K$ the algebra of all matrix-valued functions of the form $\tilde{G} = \tilde{a}I + \tilde{b}\tilde{R}$ where $\tilde{a}, \tilde{b} \in C^\alpha(\dot{\mathbb{R}})$ and

$$\tilde{R} = \begin{bmatrix} 0 & 1 \\ \tilde{\rho}^2 & 0 \end{bmatrix}, \qquad \tilde{\rho}^2(\xi) = \frac{(\xi + i)(\xi + 2i)}{(\xi - i)(\xi - 2i)}. \tag{4.2}$$

In other words, we denote by $\tilde{A}_K$ the class studied in the previous section. As we will show next, the elements in the two algebras can be related by simple rational matrices. Let $M: A_K \to \tilde{A}_K$, $MG = B^{-1}GB$ where

$$B(\xi) = \begin{bmatrix} \dfrac{\xi + i}{\xi - 2i} & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.3}$$

It is obvious that the mapping M is bijective and preserves the eigenvalues. More precisely, we have:

PROPOSITION 4.1. *Let $G \in A_K$, $G = aI + bR$. Then $\tilde{G} = MG$ has the same eigenvalues as G and it is of the form $\tilde{G}\ \imath\ \tilde{a}I\ f\ \tilde{b}\tilde{R}$ where*

$$\tilde{a}(\xi) = a(\xi), \qquad \tilde{b}(\xi) = \frac{\xi - 2i}{\xi + i}\, b(\xi), \qquad \textit{for } \xi \in \mathbb{R}. \tag{4.4}$$

Proof. The first part is a direct consequence of the definition of $M: A_K \to \tilde{A}_K$. As to the second part, it can be verified by direct calculation. ∎

Moreover, M preserves the existence of a canonical factorization in the algebra, as the following theorem shows.

THEOREM 4.2. *$G \in A_K$ admits a canonical factorization in the algebra A_K iff $\tilde{G} = MG$ admits a canonical factorization in the algebra $\tilde{A}_K$. Furthermore, if $\tilde{G} = \tilde{G}_-\tilde{G}_+$ is such a factorization, $G = G_-G_+$ with $G_\pm = B\tilde{G}_\pm B^{-1}$ is a canonical factorization of G (in the algebra A_K).*

Proof. If $G = G_-G_+$ and $G_-^{\pm 1} \in A_K$, $G_+^{\pm 1} \in A_K$, then G_- is of the form $G_- = g_{1-}I + g_{2-}R$ where $g_{1-}, g_{2-} \in L_\infty^-(\mathbb{R})$, $g_{2-}(-i) = 0$ and $\det G_- = g_{1-}^2 - \rho^2 g_{2-}^2 \in \mathcal{G}L_\infty^-(\mathbb{R})$. Thus, according to Proposition 4.1, $\tilde{G}_- = B^{-1}G_-B = g_{1-}I + \tilde{g}_{2-}\tilde{R}$ where $\tilde{g}_{2-} \in L_\infty^-(\mathbb{R})$ since

$$\tilde{g}_{2-}(\xi) = \frac{\xi - 2i}{\xi + i}\, g_{2-}(\xi)$$

and $g_{2-}(-i) = 0$. Furthermore, $\det \tilde{G}_- = \det G_- \in \mathcal{G}L_\infty^-(\mathbb{R})$.

Analogously, G_+ has the form $G_+ = g_{1+}I + g_{2+}R$ where $g_{1+}, g_{2+} \in L_\infty^+(\mathbb{R})$, $g_{2+}(i) = 0$, and $\det G_+ = g_{1+}^2 - \rho^2 g_{2+}^2 \in \mathcal{G}L_\infty^+(\mathbb{R})$. Hence $\tilde{G}_+ = B^{-1}G_+B = g_{1+}I + \tilde{g}_{2+}\tilde{R}$ where $\tilde{\rho}^2\tilde{g}_{2+} \in L_\infty^+(\mathbb{R})$ since

$$(\tilde{\rho}^2\tilde{g}_{2+})(\xi) = \frac{\xi + 2i}{\xi - i}\, g_{2+}(\xi)$$

and $g_{2+}(i) = 0$. Moreover, $\det \tilde{G}_+ = \det G_+ \in \mathcal{G}L_\infty^+(\mathbb{R})$.

Therefore $\tilde{G} = \tilde{G}_-\tilde{G}_+$ is a canonical generalized factorization of $\tilde{G}$ such that $\tilde{G}_-^{\pm 1}, \tilde{G}_+^{\pm 1} \in \tilde{A}_K$.

The converse is proved in a similar way. ∎

Remark 4.3. The last theorem shows that if G has eigenvalues d_1 and d_2 and $\tilde{G} = MG$, then both matrices admit a canonical factorization in the corresponding algebras iff condition (3.8) is satisfied.

Assuming now that condition (3.8) is not satisfied and therefore G has a generalized factorization whose factors are not in the algebra A_K, this factorization can be obtained as follows.

Let $\tilde{G} = MG$. According to what was established at the end of Section 3, there is a rational matrix $\tilde{Q}$ such that $\tilde{Q}\tilde{G}$ has a canonical factorization in the algebra $\tilde{A}_K$. Therefore $G_1 = B\tilde{Q}\tilde{G}B^{-1}$ has a canonical factorization in the algebra A_K (cf. Theorem 4.2). Let it be $G_1 = G_{1-}^A G_{1+}^A$.

Since we can write $G_1 = QG$ with $Q = B\tilde{Q}B^{-1}$, it is clear that a generalized factorization for G can be obtained from $G = G_{1-}^A Q^{-1} G_{1+}^A$.

5. Example

In this section we present an example to which the results established in the preceding sections can be applied. In fact we verify that G satisfies the necessary and sufficient condition for existence of a canonical factorization and give the corresponding factors (which do not belong to the same algebra A_K as G). Let

$$G = \begin{bmatrix} 1 & i\rho^{-1} \\ i\rho & 1 \end{bmatrix}, \tag{5.1}$$

where ρ^2 is defined by (3.1). G has the eigenvalues $d_1 = 1 + i$, $d_2 = 1 - i$ and it is obvious that

$$\int_{\mathbb{R}} \frac{\log(\pm d_1/d_2)(\xi)}{\sqrt{(\xi^2 + 1)(\xi^2 + 4)}}\, d\xi = \log(\pm i) \int_{\mathbb{R}} \frac{d\xi}{\sqrt{(\xi^2 + 1)(\xi^2 + 4)}} \neq 0. \tag{5.2}$$

Therefore we conclude that G admits a canonical generalized factorization (cf. Theorem 2.1) whose factors do not belong to the algebra A_K (cf. Theorem 3.2).

On the other hand we have $\varepsilon = \log(-i) = -i\pi/2$ (cf. (3.6)) and $\theta = K - \varepsilon K'/\pi = K + iK'/2 \in R_s$ (cf. (3.26), (3.27)), which means that in this case we have $n = 1$, $k = 0$. Thus (see Lemma 3.3) the value of μ satisfying (3.28) is given by (3.29). Since (cf. [1])

$$\operatorname{sn}\left(K + i\frac{K'}{2}\right) = \sqrt{2} \tag{5.3}$$

we have $z_0 = -i\sqrt{2}$ which yields $\mu = -i$.

Following the notation of Section 3 we obtain, for the first column of r_-G_- and $r_+G_+^{-1}$,

$$A_1 = \frac{-\mu}{3}\, \rho_-(-i)\, \sqrt{(d'_{1-}d'_{2-})^{-1}_{(-i)}}, \qquad A_0 = -A_1\,\frac{z_0 - i}{z_0 + i}, \tag{5.4}$$

$$\phi_1^- = C_1 r_- \sqrt{\frac{\xi - 2i}{\xi - 2/z_0}} \frac{\sqrt{d_{1-}d_{2-}}}{\sqrt{1-\mu^2}} \left[\rho_+ \operatorname{sh}\left(\frac{1}{2} r_+^{-1}\rho^{-1}F^-\right) - \mu\rho_-^{-1} \operatorname{ch}\left(\frac{1}{2} r_+^{-1}\rho^{-1}F^-\right)\right]$$

$$\phi_2^- = C_1 r^{-1} r_- \sqrt{\frac{\xi - 2i}{\xi - 2/z_0}} \frac{\sqrt{d_{1-}d_{2-}}}{\sqrt{1-\mu^2}} \left[\rho_-^{-1} \operatorname{ch}\left(\frac{1}{2} r_+^{-1}\rho^{-1}F^-\right) - \mu\rho_-^{-2}\rho_+^{-1} \operatorname{sh}\left(\frac{1}{2} r_+^{-1}\rho^{-1}F^-\right)\right]$$

$$\phi_1^+ = C_1 r_+ \sqrt{\frac{\xi - z_0}{\xi + 2i}} \sqrt{-d_{1+}^{-1}d_{2+}^{-1}} \operatorname{ch}\left(\frac{1}{2} r_+^{-1}\rho^{-1}F^+\right)$$

$$\phi_2^+ = -C_1 \sqrt{\frac{\xi - z_0}{\xi + i}} \frac{\sqrt{-d_{1+}^{-1}d_{2+}^{-1}}}{\sqrt{(\xi - i)(\xi - 2i)}} \operatorname{sh}\left(\frac{1}{2} r_+^{-1}\rho^{-1}F^+\right),$$

where (for A_1 given by (5.4))

$$C_1 = \frac{2iA_1}{z_0 + i}, \qquad F^\pm = P^\pm F, \qquad F = r_+\rho \log\left(-\frac{d_1'}{d_2'}\right). \tag{5.5}$$

As to the second column of r_-G_- and $r_+G_+^{-1}$, we obtain

$$A_0 = \frac{\mu}{3}\rho_+^{-1}(i)\sqrt{-(d_{1+}''d_{2+}'')(i)}, \qquad A_1 = -A_0 \frac{z_0 - i}{z_0 + i},$$

$$\phi_3^- = C_2 \sqrt{\frac{\xi + z_0}{\xi - i}} \frac{\sqrt{d_{1-}d_{2-}}}{\sqrt{(\xi + i)(\xi + 2i)}} \operatorname{sh}\left(\frac{1}{2} r_+^{-1}\rho^{-1}\tilde{F}^-\right)$$

$$\phi_4^- = C_2 r_- \sqrt{\frac{\xi + z_0}{\xi - 2i}} \sqrt{d_{1-}d_{2-}} \operatorname{ch}\left(\frac{1}{2} r_+^{-1}\rho^{-1}\tilde{F}^-\right)$$

$$\phi_3^+ = C_2 r_+ \sqrt{\frac{\xi + 2i}{\xi + 2/z_0}} \frac{\sqrt{-d_{1+}^{-1}d_{2+}^{-1}}}{\sqrt{1-\mu^2}} \left[\rho_+ r \operatorname{ch}\left(\frac{1}{2} r_+^{-1}\rho^{-1}\tilde{F}^+\right) - \mu r^{-1}\rho^{-2}\rho_-^{-1} \operatorname{sh}\left(\frac{1}{2} r_+^{-1}\rho^{-1}\tilde{F}^+\right)\right] \tag{5.6}$$

$$\phi_4^+ = -C_2 r_+ \sqrt{\frac{\xi + 2i}{\xi + 2/z_0}} \frac{\sqrt{-d_{1+}^{-1}d_{2+}^{-1}}}{\sqrt{1-\mu^2}} \left[\rho_-^{-1} \operatorname{sh}\left(\frac{1}{2} r_+^{-1}\rho^{-1}\tilde{F}^+\right) - \mu\rho_+ \operatorname{ch}\left(\frac{1}{2} r_+^{-1}\rho^{-1}\tilde{F}^+\right)\right],$$

where (for A_0 given by (5.6))

$$C_2 = \frac{2iA_0}{z_0 + i}, \qquad \tilde{F}^{\pm} = P^{\pm}\tilde{F}, \qquad \tilde{F} = r_+\rho \log\left(-\frac{d_1''}{d_2''}\right). \tag{5.7}$$

Remark 5.1. The values of A_1 in (5.4) and A_0 in (5.6) depend only on the normalization of the generalized factorization of G. Therefore they are actually arbitrary (as well as the values of C_1 and C_2 defined in (5.5) and (5.7)).

These results (which are valid in general for $n = 1$) can be considerably simplified in our case and be expressed only in terms of algebraic functions. This is made possible by the fact that $\mu^2 = -1$ (or, in terms of the matrix function $G = aI + bR$, because $a^2 = -b^2\rho^2$). In fact we have

$$\text{(i)} \quad F = r_+\rho \log\left(-i\,\frac{1 - i\rho r}{1 + i\rho r}\right) = F^- + F^+ \tag{5.8}$$

with

$$F^+ = 2r_+\rho \log\left(\frac{1+i}{2}\sqrt{1 + \frac{q_-}{c(\xi + i\sqrt{2})}} + \frac{1-i}{2}\sqrt{1 - \frac{q_-}{c(\xi + i\sqrt{2})}}\right)$$

$$F^- = 2r_+\rho \log\left[\sqrt{-i\,\frac{1 - i\rho r}{1 + i\rho r}}\left(\frac{1-i}{2}\sqrt{1 + \frac{q_-}{c(\xi + i\sqrt{2})}} + \frac{1+i}{2}\sqrt{1 - \frac{q_-}{c(\xi + i\sqrt{2})}}\right)\right]$$

$$c = -\sqrt{6}\,(\sqrt{2} + 1), \qquad q_-(\xi) = \sqrt{(\xi - i)(\xi - 2i)} \tag{5.9}$$

$$\text{(ii)} \quad \tilde{F} = r_+\rho \log\left(i\,\frac{1 + i\rho r}{1 - i\rho r}\right) = \tilde{F}^- + \tilde{F}^+ \tag{5.10}$$

with

$$\tilde{F}^+ = 2r_+\rho \log\left[\sqrt{i\,\frac{1 + i\rho r}{1 - i\rho r}}\left(\frac{1-i}{2}\sqrt{1 - \frac{q_+}{\bar{c}(\xi - i\sqrt{2})}} - \frac{1+i}{2}\sqrt{1 + \frac{q_+}{\bar{c}(\xi - i\sqrt{2})}}\right)\right]$$

$$\tilde{F}^- = 2r_+\rho \log\left(\frac{1+i}{2}\sqrt{1 - \frac{q_+}{\bar{c}(\xi - i\sqrt{2})}} + \frac{1-i}{2}\sqrt{1 + \frac{q_+}{\bar{c}(\xi - i\sqrt{2})}}\right)$$

$$\bar{c} = -c = \sqrt{6}\,(\sqrt{2} + 1), \qquad q_+(\xi) = \sqrt{(\xi + i)(\xi + 2i)}. \tag{5.11}$$

Remark 5.2. The value of c given in (5.9) is such that

$$\begin{aligned} &[c(\xi + i\sqrt{2}) + q_-(\xi)]_{\xi=-2i} = 0, \qquad &&[c(\xi + i\sqrt{2}) - q_-(\xi)]_{\xi=-i} = 0 \\ &[\bar{c}(\xi - i\sqrt{2}) + q_+(\xi)]_{\xi=i} = 0, \qquad &&[\bar{c}(\xi - i\sqrt{2}) - q_+(\xi)]_{\xi=2i} = 0. \end{aligned} \tag{5.12}$$

This yields for the first columns

$$\phi_1^+ = \tilde{C}_1 r_+ \sqrt{\frac{\xi + i\sqrt{2}}{\xi + 2i}} \left(\sqrt{1 + \frac{q_-}{c(\xi + i\sqrt{2})}} + \sqrt{1 - \frac{q_-}{c(\xi + i\sqrt{2})}} \right),$$

$$\phi_2^+ = -i\tilde{C}_1 r_+ \rho \sqrt{\frac{\xi + i\sqrt{2}}{\xi + 2i}} \left(\sqrt{1 + \frac{q_-}{c(\xi + i\sqrt{2})}} - \sqrt{1 - \frac{q_-}{c(\xi + i\sqrt{2})}} \right),$$

$$\phi_1^- = 2\tilde{C}_1 r_- \sqrt{\frac{\xi + i\sqrt{2}}{\xi + 2i}} \sqrt{1 + \frac{q_-}{c(\xi + i\sqrt{2})}},$$

$$\phi_2^- = 2i\tilde{C}_1 r_- \frac{\sqrt{(\xi + i)(\xi + i\sqrt{2})}}{q_-} \sqrt{1 - \frac{q_-}{c(\xi + i\sqrt{2})}},$$

and for the second columns

$$\phi_3^+ = -i\tilde{C}_2 r_+ \frac{\sqrt{(\xi - i)(\xi - i\sqrt{2})}}{q_+} \sqrt{1 + \frac{q_+}{\bar{c}(\xi - i\sqrt{2})}}, \tag{5.13}$$

$$\phi_4^+ = \tilde{C}_2 r_+ \sqrt{\frac{\xi - i\sqrt{2}}{\xi - 2i}} \sqrt{1 - \frac{q_+}{\bar{c}(\xi - i\sqrt{2})}},$$

$$\phi_3^- = -i\tilde{C}_2 q_+^{-1} \sqrt{\frac{\xi - i\sqrt{2}}{\xi - i}} \left[\sqrt{1 + \frac{q_+}{\bar{c}(\xi - i\sqrt{2})}} - \sqrt{1 - \frac{q_+}{\bar{c}(\xi - i\sqrt{2})}} \right],$$

$$\phi_4^- = \tilde{C}_2 r_- \sqrt{\frac{\xi - i\sqrt{2}}{\xi - 2i}} \left[\sqrt{1 + \frac{q_+}{\bar{c}(\xi - i\sqrt{2})}} + \sqrt{1 - \frac{q_+}{\bar{c}(\xi - i\sqrt{2})}} \right],$$

with q_+, q_-, c, $\bar{c}$ defined in (5.9) and (5.11). It can be verified by direct calculation that this indeed provides a generalized factorization of G defined in (5.1).

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